Commun. Combin., Cryptogr. & Computer Sci., 1 (2021), 89–91

Communications in Combinatorics, Cryptography & Computer Science

Journal Homepage: www.vonneumann-publishing.com/cccs

On the tree-number of conjugacy class graphs of some metacyclic groups

Zeinab Foruzanfar[a]

[a] *Imam Khomeini International University - Buin Zahra Higher Education Center of Engineering and Technology, Qazvin, Iran.*

Abstract

For a finite group G with $V(G)$ as the set of all non-central conjugacy classes of it, the conjugacy class graph $\Gamma(G)$ is defined as: its vertex set is the set $V(G)$ and two distinct vertices a^G and b^G are connected with an edge if $(o(a), o(b)) > 1$. In this paper, we determine the tree-number of the conjugacy class graphs of metacyclic groups of order less than thirty.

Keywords: Metacyclic group, conjugacy class graph, tree-number.

2020 MSC: 20E45, 05C50, 20D60.

1. Introduction

A graph Γ is a pair (V, E), where V is a set whose elements are called vertices and E is a set of paired vertices, whose elements are called edges. Suppose that Γ be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. The adjacency matrix of Γ denoted by A, is an $n \times n$ matrix whose entries a_{ij} are 1, when v_i and v_j are adjacent and 0 otherwise. The degree of a vertex v_i is denoted by $deg(v_i)$ and the degree matrix denoted by Δ is defined as $\Delta = diag(deg(v_1), deg(v_2), \ldots, deg(v_n))$, which is a diagonal matrix. Then, the Laplacian matrix of Γ is denoted by Q which satisfies $Q = \Delta - A$. We denote by $\mu_0, \mu_1, \ldots, \mu_n$ the eigenvalues of the Laplacian matrix of Γ. It is proved in [3] that at least one of these eigenvalues is zero. Without loss of generality, we assume that $\mu_0 = 0$. The characteristic polynomial of the Laplacian matrix Q is $\sigma(\Gamma, \mu) = det(\mu I - Q)$. The tree-number of Γ is the number of spanning trees of Γ and is denoted by $\kappa(\Gamma)$. For disconnected graphs Γ, $\kappa(\Gamma)$ is defined 0 (see [3]).

Let G be a finite non-abelian group and $V(G)$ be the set of all non-central conjugacy classes of G. A conjugacy class graph $\Gamma(G)$ according to the orders of representatives of conjugacy classes is defined in [6] as below: its vertex set is the set $V(G)$ and two distinct vertices a^G and b^G are connected with an edge if $(o(a), o(b)) > 1$. A metacyclic group is an extension of a cyclic group by a cyclic group. Equivalently, a metacyclic group is a group G having a cyclic normal subgroup N, such that the quotient $\frac{G}{N}$ is also cyclic. Clearly, every cyclic group is metacyclic. There are some known results about metacyclic groups which we point them briefly. One can find the proofs in [4] and [5]. The subgroups and quotient groups of metacyclic groups are metacyclic. The direct product or semidirect product of two cyclic groups is metacyclic. So, the dihedral groups and the semi-dihedral groups are metacyclic. The dicyclic groups are metacyclic. Every finite group of squarefree order is metacyclic. Recall that $K \rtimes H$ is a semidirect product of K and H with normal subgroup K and $K \rtimes_f H$ is the Frobenius group with kernel K and complement H. All further unexplained notations are standard. In this paper, we compute the tree-number of conjugacy class graphs of metacyclic groups of order less than thirty.

Email address: zforouzanfar@gmail.com, z.forozanfar@bzeng.ikiu.ac.ir (Zeinab Foruzanfar)

Received: November, 1, 2021 Revised: November, 15, 2021 Accepted: November, 20, 2021

2. Examples and Preliminaries

In this section, we give some examples and preliminary results that will be used in the proof of our main results.

Proposition 2.1. *([2]) The multiplicity of 0 as an eigenvalue of* Q *is equal to the number of connected components of the graph.*

Proposition 2.2. *([1]) The Laplacian matrix of the complete graph* K_n *has eigenvalues 0 with multiplicity 1 and* n *with multiplicity* $n-1$.

Corollary 2.3. *(Corollary 6.5 of [3]) Let* $0 \leqslant \mu_1 \leqslant \ldots \leqslant \mu_{n-1}$ *be the Laplacian spectrum of a graph* Γ *with* n *vertices. Then* $\kappa(\Gamma) = \frac{\mu_1\mu_2\ldots\mu_{n-1}}{n}$.

In the following we present some examples of metacyclic groups and find their characteristic Laplacian polynomials and eigenvalues.

Example 2.4. Since $\mathbb{Z}_4$ is a cyclic normal subgroup of Q_8 such that $|\frac{Q_8}{\mathbb{Z}_4}| = 2$, we deduce that Q_8 is metacyclic. Also, we have $\sigma(\Gamma,\mu) = \mu(\mu-3)^2$.

Example 2.5. Since $\mathbb{Z}_8$ is a cyclic normal subgroup of Q_{16} such that $|\frac{Q_{16}}{\mathbb{Z}_8}| = 2$, we deduce that Q_{16} is metacyclic. Also, we have $\sigma(\Gamma,\mu) = \mu(\mu-5)^4$.

Example 2.6. Since $\mathbb{Z}_8$ is a cyclic normal subgroup of $M_4(2)$ such that $|\frac{M_4(2)}{\mathbb{Z}_8}| = 2$, we deduce that $M_4(2)$ is metacyclic. Also, we have $\sigma(\Gamma,\mu) = \mu(\mu-6)^5$.

Example 2.7. Since $\mathbb{Z}_6$ is a cyclic normal subgroup of $\mathbb{Z}_3 \times S_3$ such that $|\frac{\mathbb{Z}_3\times S_3}{\mathbb{Z}_6}| = 3$, we deduce that $\mathbb{Z}_3 \times S_3$ is metacyclic. Also, we have $\sigma(\Gamma,\mu) = \mu(\mu-2)(\mu-5)^2(\mu-6)^2$.

Example 2.8. Since $\mathbb{Z}_{12}$ is a cyclic normal subgroup of $\mathbb{Z}_4 \times S_3$ such that $|\frac{\mathbb{Z}_4\times S_3}{\mathbb{Z}_{12}}| = 2$, we deduce that $\mathbb{Z}_4 \times S_3$ is metacyclic. Also, we have $\sigma(\Gamma,\mu) = \mu(\mu-3)(\mu-7)^3(\mu-8)^3$.

Example 2.9. Since $\mathbb{Z}_{12}$ is a cyclic normal subgroup of $\mathbb{Z}_3 \times D_8$ such that $|\frac{\mathbb{Z}_3\times D_8}{\mathbb{Z}_{12}}| = 2$, we deduce that $\mathbb{Z}_3 \times D_8$ is metacyclic. Also, we have $\sigma(\Gamma,\mu) = \mu(\mu-9)^8$.

Example 2.10. Since $\mathbb{Z}_{12}$ is a cyclic normal subgroup of $\mathbb{Z}_3 \times Q_8$ such that $|\frac{\mathbb{Z}_3\times Q_8}{\mathbb{Z}_{12}}| = 2$, we deduce that $\mathbb{Z}_3 \times Q_8$ is metacyclic. Also, we have $\sigma(\Gamma,\mu) = \mu(\mu-9)^8$.

Example 2.11. Since $\mathbb{Z}_6$ is a cyclic normal subgroup of $\mathbb{Z}_2 \times Dic_3$ such that $\frac{\mathbb{Z}_2\times Dic_3}{\mathbb{Z}_6} \cong \mathbb{Z}_4$, we deduce that $\mathbb{Z}_2 \times Dic_3$ is metacyclic. Also, we have $\sigma(\Gamma,\mu) = \mu(\mu-3)(\mu-7)^3(\mu-8)^3$.

Example 2.12. Since $\mathbb{Z}_9$ is a cyclic normal subgroup of $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_3$ such that $|\frac{(\mathbb{Z}_3)^2\rtimes\mathbb{Z}_3}{\mathbb{Z}_9}| = 2$, we deduce that $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_3$ is metacyclic. Also, we have $\sigma(\Gamma,\mu) = \mu(\mu-8)^7$.

3. Main results

In this section, we give our main result as follows.

Theorem 3.1. *Let* G *be a non-abelian metacyclic group of order least than 30 and* $\Phi = (|g_1^G|, |g_2^G|, \ldots, |g_n^G|)$*, such that* g_i^G *are the conjugacy classes of* G *for* $1 \leqslant i \leqslant n$*. Then the values of tree-numbers of* $\Gamma(G)$ *is given in Table 1.*

Proof. Since the direct product and semidirect product of two cyclic groups, the dihedral groups, the semi-dihedral groups and the dicyclic groups are metacyclic, we deduce that S_3, D_8, D_{10}, D_{12}, Dic_3, D_{14}, D_{16}, SD_{16}, $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$, D_{18}, D_{20}, $\mathbb{Z}_5 \rtimes_f \mathbb{Z}_4$, Dic_5, $\mathbb{Z}_7 \rtimes_f \mathbb{Z}_3$, D_{22}, D_{24}, Dic_6, $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$, D_{26}, Dic_7 and D_{28} are metacyclic groups. So, the list of metacyclic groups of order less than 30 be deduced from checking the groups of order less than 30 and the examples mentioned in Section 2. Now, by the orders of representatives of conjugacy classes of these groups, their Laplacian matrices and the associated eigenvalues are obtained. So, the tree-numbers of the conjugacy class graphs of these groups can be computed by the formula $\kappa(\Gamma) = \frac{\mu_1\mu_2\ldots\mu_{n-1}}{n}$, as is shown in the Table 1. □

Table 1: Tree number of metacyclic groups of order less than 30

G	Φ	Orders of representatives of conjugacy classes of G	$\kappa(\Gamma)$
S_3	$(1,3,2)$	$(1,2,3)$	0
Q_8	$(1,2,2,1,2)$	$(1,4,4,2,4)$	3
D_8	$(1,2,2,1,2)$	$(1,2,4,2,2)$	3
D_{10}	$(1,5,2,2)$	$(1,2,5,5)$	0
D_{12}	$(1,3,2,2,3,1)$	$(1,2,6,3,2,2)$	3
Dic_3	$(1,3,1,2,3,2)$	$(1,4,2,3,4,6)$	3
D_{14}	$(1,7,2,2,2)$	$(1,2,7,7,7)$	0
D_{16}	$(1,4,2,2,1,4,2)$	$(1,2,8,4,2,2,8)$	125
Q_{16}	$(1,4,2,2,1,4,2)$	$(1,4,8,4,2,4,8)$	125
SD_{16}	$(1,4,4,2,1,2,2)$	$(1,4,2,4,2,8,8)$	125
$M_4(2)$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,8,2,4,2,8,8,4,4,8)$	1296
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,4,4,2,2,4,4,4,2,4)$	1296
D_{18}	$(1,9,2,2,2,2)$	$(1,2,9,3,9,9)$	0
$\mathbb{Z}_3 \times S_3$	$(1,3,1,2,3,1,2,3,2)$	$(1,2,3,3,6,3,3,6,3)$	300
D_{20}	$(1,5,1,2,5,2,2,2)$	$(1,2,2,5,2,10,5,10)$	192
$\mathbb{Z}_5 \rtimes_f \mathbb{Z}_4$	$(1,5,5,4,5)$	$(1,4,2,5,4)$	0
Dic_5	$(1,5,1,2,5,2,2,2)$	$(1,4,2,5,4,10,5,10)$	192
$\mathbb{Z}_7 \rtimes_f \mathbb{Z}_3$	$(1,7,3,7,3)$	$(1,3,7,3,7)$	0
D_{22}	$(1,11,2,2,2,2,2)$	$(1,2,11,11,11,11,11)$	0
D_{24}	$(1,6,2,1,2,6,2,2,2)$	$(1,2,4,2,3,2,12,6,12)$	5292
Dic_6	$(1,6,2,1,2,6,2,2,2)$	$(1,4,4,2,3,4,12,6,12)$	5292
$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,8,4,2,3,8,8,4,12,6,8,12)$	65856
$\mathbb{Z}_4 \times S_3$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,2,4,2,3,4,2,4,12,6,4,12)$	65856
$\mathbb{Z}_3 \times D_8$	$(1,2,2,1,1,2,2,2,1,1,2,2,2,1,2)$	$(1,2,2,3,2,4,6,6,3,6,12,6,6,6,12)$	4782969
$\mathbb{Z}_3 \times Q_8$	$(1,2,2,1,1,2,2,2,1,1,2,2,2,1,2)$	$(1,4,4,3,2,4,12,12,3,6,12,12,12,6,12)$	4782969
$\mathbb{Z}_2 \times Dic_3$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,4,2,2,3,4,4,2,6,6,4,6)$	65856
D_{26}	$(1,13,2,2,2,2,2,2,)$	$(1,2,13,13,13,13,13,13)$	0
$(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_3$	$(1,3,3,1,3,3,3,1,3,3,3)$	$(1,3,3,3,3,3,3,3,3,3,3)$	262144
Dic_7	$(1,7,1,2,7,2,2,2,2,2)$	$(1,4,2,7,4,14,7,14,7,14)$	34560
D_{28}	$(1,7,1,2,7,2,2,2,2,2)$	$(1,2,2,7,2,14,7,14,7,14)$	34560

References

[1] R. B. Bapat, *Graphs and Matrices*, Springer, New York, (2010). 2.2

[2] L. W. Beineke, R. J. Wilson, *Topics in algebraic graph theory*, **102**, Cambridge University Press, New York, (2004). 2.1

[3] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, Second edition, (1993). 1, 2.3

[4] C. E. Hempel, *Metacyclic Groups*, Commun. Algebra, **28** (2000), 3865–3897. 1

[5] J. S. Rose, *A Course on Group Theory*, New York: Dover, (1994). 1

[6] X. You, G. Qian, *A new graph related to conjugacy classes of finite groups*, Chinese Ann. Math. Ser. A, **28(5)** (2007), 631–636. 1